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Infine Functions, Nonsmooth Alternative Theorems and Vector Optimization Problems

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Abstract. In this paper we introduce a new notion of infine nonsmooth functions and give several characterizations of infineness property. We prove alternative theorems with mixed constraints (i.e., inequality and equality constraints) being described by invex-infine nonsmooth functions. We establish a necessary and sufficient condition for a solution of a vector optimization problem involving mixed constraints to be a properly efficient solution.

1. Introduction

During the last 20 years invexity is known as a concept which is a geralization of the convexity property and which can be used to extend the sufficiency of the Kuhn–Tucker conditions and the duality theory of the class of convex programs to a more general class of optimization problems. This invexity idea was first introduced by Hanson [14] for differentiable functions and was generalized to nonsmooth functions [8, 20] and multifunctions [12, 22, 23]. Invexity was also weakened in order that it can be served as a necessary optimality condition [15, 16, 24] or a characterization of problems where every Kuhn–Tucker point is a global minimizer [18, 25]. Invex functions are also useful for alternative theorems [2].

It is worth noticing that the proof of sufficiency of the Kuhn–Tucker conditions is based on the fact that the invexity of constraint functions g_j implies that of $\lambda_j g_j$ where λ_j are Kuhn–Tucker multipliers associated to g_j. This fact is true for inequality constraints since in this case all λ_j are nonnegative. Unfortunately, it fails to hold for equality constraints since λ_j are not necessarily nonnegative. Thus the usual invexity notion is suitable for optimization problems with inequality constraints, but it is not useful for problems with equality constraints. The aims of this paper are:

1. To introduce a class of locally Lipschitz functions, called the class of infine functions, which is a subclass of invex functions but which is appropriate

for optimization problems with equality constraints. (The reason for using the terminology "infine functions" is given in Remark 3.2 of Section 3.)

2. To prove alternative theorems for systems involving mixed constraints: a geometric constraint $x \in S$ (S being a closed convex subset which may not coincide with the whole space $\mathbb{R}^n$) and several inequality and equality constraints given by nonsmooth invex and infine functions.
3. To show that alternative theorems, and the invexity and infineness ideas can be applied to finding properly efficient points of nonsmooth problems of vector optimization involving mixed constraints.

The organization of this paper is as follows: in Section 2 some concepts and facts from Nonsmooth Analysis are collected. In Section 3 a new concept of infine functions on S at $x_0 \in S$ is introduced. Several sufficient conditions for infineness are given. It is shown that all they are equivalent conditions and become necessary conditions for infineness if the Clarke tangent cone of S at x_0 is a subspace. Examples of infine functions are provided. In Section 4 two alternative theorems are given for locally Lipschitz vector-valued maps f, g and h with suitable invexity and infineness properties. The first one deals with system

$$g(x) \leqq 0, h(x) = 0, x \in S \tag{1.1}$$

and the second one differs from the first in that a strict inequality $f(x) < 0$ is added to system (1.1). The second alternative theorem includes as a special case a known result [2] where h is absent. Our proof is simpler and quite different from that of [2]. An alternative theorem is applied to characterizing properly efficient points of a vector optimization problem of map f subject to constraints (1.1).

2. Preliminaries

Let $\mathbb{R}^n$ be an Euclidean space. For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ we will use the following notation:

$$\begin{aligned}
x = y &\Leftrightarrow x_i = y_i, \quad \text{for all } i;\\
x < y &\Leftrightarrow x_i < y_i, \quad \text{for all } i;\\
x \leqq y &\Leftrightarrow x_i \leqq y_i, \quad \text{for all } i;\\
x \leqslant y &\Leftrightarrow x \leqq y \quad \text{and } x \neq y.
\end{aligned}$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, that is, for any $z \in \mathbb{R}^n$, there exist $\alpha > 0, \beta > 0$ such that for any $x, x' \in \mathbb{R}^n$ with $\| x - z \| < \alpha$, $\| x' - z \| < \alpha$, $| f(x) - f(x') | \leqq \beta \| x - x' \|$, and let $x_0 \in \mathbb{R}^n$. Then the Clarke directional derivative of f at x_0 in the direction v is defined by

$$f^0(x_0, v) = \limsup_{y \to x_0\, \lambda \downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

and the Clarke subdifferential of f at x_0 is defined by

$$\partial f(x_0) = \{\xi \in \mathbb{R}^n : f^0(x_0, v) \geqq \langle \xi, v \rangle \ \forall v \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

It is well known [4] that for any $v \in \mathbb{R}^n$

$$f^0(x_0, v) = \max_{\xi \in \partial f(x_0)} \langle \xi, v \rangle \tag{2.1}$$

and $\partial f(x_0)$ is a nonempty compact convex subset of $\mathbb{R}^n$. Also,

$$-\partial f(x_0) = \partial(-f)(x_0). \tag{2.2}$$

Let S be a closed subset of $\mathbb{R}^n$ and $x_0 \in S$. The Clarke [4] tangent cone to S at x_0 is defined by

$$T_S(x_0) := \{v \in \mathbb{R}^n : d_S^0(x_0, v) = 0\}, \tag{2.3}$$

where $d_S(x) = \inf_{z \in S} \parallel z - x \parallel$, and the Clarke [4] normal cone to S at x_0 is defined by

$$N_S(x_0) := \{w \in \mathbb{R}^n : \langle v, w \rangle \leqq 0 \ \forall v \in T_S(x_0)\}. \tag{2.4}$$

A subset $A \subset R^n$ is said to be a cone if $\lambda x \in A$ for all $x \in A$ and $\lambda \geqq 0$. A cone which is a convex set is said to be a convex cone. For any nonempty subset $A \subset R^n$ denote by *cone* A the intersection of all *convex* cones containing A. It is easy to check that *cone* A is a convex cone consisting of all points of the form $\sum_{i=1}^m \lambda_i x_i$ where m is a positive integer, $x_i \in A$ and $\lambda_i \geqq 0$. Also, $cone\, A = cone\,(co\, A)$ where $co\, A$ stands for the convex hull of A. When A is a convex set, $cone\, A = \{\lambda x : \lambda \geqq 0,\ x \in A\}$. It is proved in [4] that

$$N_S(x_0) = cl\; cone\; \partial d_S(x_0) \tag{2.5}$$

where $cl\, A$ denotes the closure of A. For any nonempty subset $A \subset \mathbb{R}^n$ denote by aff$A$ the affine hull of A. This is the intersection of all affine sets containing the set A. It is known [21] that aff$A$ consists of all points of the form $\sum_{i=1}^m \lambda_i x_i$ with $\sum_{i=1}^m \lambda_i = 1$ where m is a positive integer, $x_i \in A$ and $\lambda_i \in \mathbb{R}$ (λ_i may not be nonnegative).

For each $i = 1, 2, ..., m$ let A_i be a nonempty compact convex set of $\mathbb{R}^n$ and β_i a real number. Then $A_i \times \{-\beta_i\}$ is a subset of $\mathbb{R}^n \times \mathbb{R}$. Let

$$\varphi_i(t) = \max_{\xi \in A_i} \langle \xi, t \rangle. \tag{2.6}$$

PROPOSITION 2.1. *System*

$$\varphi_i(t) < 0 \qquad (i = 1, 2, ..., m'), \tag{2.7}$$

$$\varphi_i(t) \leqq 0 \qquad (i = m' + 1, m' + 2, ..., m) \tag{2.8}$$

has a solution $t \in R^n$ if and only if

$$0 \notin \operatorname{co} \bigcup_{i=1}^{m'} A_i + \operatorname{cl} \operatorname{cone} \bigcup_{i=m'+1}^{m} A_i . \tag{2.9}$$

PROPOSITION 2.2. *System*

$$\varphi_i(t) \leqq \beta_i , \; i = 1, 2, ..., m, \tag{2.10}$$

has a solution $t \in R^n$ if and only if

$$(0, 1) \notin \operatorname{cl} \operatorname{cone} \bigcup_{i=1}^{m} \left[A_i \times \{-\beta_i\}\right] \tag{2.11}$$

(0 being the origin of $\mathbb{R}^n$).

The proof of Propositions 2.1 and 2.2 can be found in [24, 25]. For reader's convenience let us give a sketch of this proof. We start by Proposition 2.1. If (2.9) holds then the intersection of the compact convex set

$$-\operatorname{co} \bigcup_{i=0}^{m'} A_i$$

and the closed convex set

$$\operatorname{cl} \operatorname{cone} \bigcup_{i=m'+1}^{m} A_i$$

is empty. Using a separation theorem we can find a vector $t \in \mathbb{R}^n$ which is a solution of system (2.7), (2.8). Conversely, if the last system has a solution, then taking account of formulas (2.6) we see that the above intersection is empty. In other words, condition (2.9) holds.

Proposition 2.2 can be deduced from Proposition 2.1. This is possible since the consistency of the nonhomogeneous system (2.10) is equivalent to that of a homogeneous system of the kind (2.7), (2.8) where $m' = 1$, $A'_1 = \{(0, -1)\}$ and $A'_i = A_i \times \{-\beta_i\}$ for $i \neq 1$, and the variable $\xi \in \mathbb{R}^n$ is replaced by an extended variable $\xi' = (\xi, r) \in \mathbb{R}^n \times \mathbb{R}$.

We conclude this section by an elementary result which will be needed later on.

Let H be a nonempty compact convex set of $\mathbb{R}^n$ and β a real number.

PROPOSITION 2.3. *Let*

$$\widehat{E} = \operatorname{cone}\Big[\Big(\bigcup_{i=1}^{m} \left[A_i \times \{-\beta_i\}\right]\Big) \bigcup \big(H \times \{-\beta\}\big)\Big], \tag{2.12}$$

$$E = \bigcup_{a \in A} \operatorname{cl} \operatorname{cone}\Big[\Big(\bigcup_{i=1}^{m} \left[\{a_i\} \times \{-\beta_i\}\right]\Big) \bigcup \big(H \times \{-\beta\}\big)\Big], \tag{2.13}$$

where $a = (a_1, a_2, \ldots a_m) \in A := A_1 \times A_2 \times \cdots \times A_m$. *Then* $cl\,\widehat{E} = cl\,E$. *If, in addition,* $\widehat{E}$ *is closed then* $\widehat{E} = E$.

Proof. Since for all $a_i \in A_i,\ i = 1, 2, \ldots, m$,

$$\text{cone}\Big[\Big(\bigcup_{i=1}^{m}\big[\{a_i\} \times \{-\beta_i\}\big]\Big)\bigcup\big(H \times \{-\beta\}\big)\Big] \subset \widehat{E}$$

we have

$$E \subset cl\,\widehat{E}. \tag{2.14}$$

On the other hand,

$$\widehat{E} = \bigcup_{a\in A}\text{cone}\Big[\Big(\bigcup_{i=1}^{m}\big[\{a_i\} \times \{-\beta_i\}\big]\Big)\bigcup\big(H \times \{-\beta\}\big)\Big] \subset E. \tag{2.15}$$

All the conclusions of Proposition 2.3 are clear from (2.14) and (2.15). □

3. Infine Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, S a closed subset of $\mathbb{R}^n$ and $x_0 \in S$. Then f is said to be *infine* on S at x_0 if for any $x \in S$ and any $\xi \in \partial f(x_0)$, there exists $\eta \in T_S(x_0)$ such that

$$f(x) - f(x_0) = \langle \xi, \eta \rangle. \tag{3.1}$$

Using (2.2) and (3.1), we see that f is infine on S at x_0 if and only if $-f$ is infine on S at the same point x_0. When $S = \mathbb{R}^n$, then $T_S(x_0) = \mathbb{R}^n$, and hence f is infine on $\mathbb{R}^n$ at x_0 if and only if for any $x \in \mathbb{R}$ and any $\xi \in \partial f(x_0)$, there exists $\eta \in \mathbb{R}^n$ such that

$$f(x) - f(x_0) = \langle \xi, \eta \rangle.$$

When f is of class C^1, then the equality (3.1) reduces to the following form:

$$f(x) - f(x_0) = f'_{x_0}\eta$$

where f'_{x_0} is the Fréchet derivative of f at x_0.

Observe that in the above definition if S is a neighbourhood of x_0 then ($T_S(x_0) = \mathbb{R}^n$ and) infineness is understood in the local sense; and if S is the whole space $\mathbb{R}^n$ then ($T_S(x_0) = \mathbb{R}^n$ and) infineness is understood in the global sense. Therefore, the introduction of the subset S in the above definition is useful since it gives a unified approach to local and global infineness. A similar situation can be found in the case of invexity [19].

Observe also that in [9, 10] Craven introduced a notion of cone-invexity of a nonsmooth vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^p$ with respect to a given cone of $\mathbb{R}^p$. For a special case where $p = 1$ and the mentioned cone coincides with the origin of the real line $\mathbb{R}$, this notion means that there exists a map $\eta : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\forall \xi \in \partial f(x_0) : f(x) - f(x_0) = \langle \xi, \eta(x) \rangle.$$

The difference between this notion of Craven and our infineness definition is that Craven requires that, for each $x \in \mathbb{R}^n$, the point $\eta(x)$ satisfying the just written equality must be the same for all $\xi \in \partial f(x_0)$ while in our infineness definition η depends not only on x but also on $\xi \in \partial f(x_0)$. Thus, the class of infine functions is larger than the class of cone-invex functions applied to the above special case. The function introduced in Example 3.1 below is infine, but it is not cone-invex.

Now we characterize infine functions.

PROPOSITION 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and S a closed subset of $\mathbb{R}^n$. Consider the following conditions:*

(a) f is infine on S at $x_0 \in S$;
(b) $0 \in \partial f(x_0) + N_S(x_0) - N_S(x_0)$ implies that f is constant on S; and
(c) $0 \in (\partial f(x_0) + N_S(x_0)) \bigcup (\partial f(x_0) - N_S(x_0))$ implies that f is constant on S.

Then $(b) \Rightarrow (c) \Rightarrow (a)$; and $(a) \Rightarrow (b)$ if $T_S(x_0)$ is a subspace of $\mathbb{R}^n$.

Proof. $(b) \Rightarrow (c)$: It is clear since $0 \in N_S(x_0)$.

$(c) \Rightarrow (a)$: If f is constant on S, f is infine on S at x_0 with respect to $\eta \equiv 0$. If f is not constant on S, then by condition (c)

$$0 \notin \partial f(x_0) + N_S(x_0) \tag{3.2}$$

and

$$0 \notin \partial f(x_0) - N_S(x_0). \tag{3.3}$$

From (3.2) and a separation theorem, there exists $t \in \mathbb{R}^n$ such that

$$0 > \max_{\xi \in \partial f(x_0)} \langle \xi, t \rangle + \sup_{y \in N_S(x_0)} \langle y, t \rangle. \tag{3.4}$$

If $t \notin T_S(x_0)$, there exists $y' \in N_S(x_0)$ such that $\langle t, y' \rangle > 0$. This means that the right-hand side of (3.4) may tend to $+\infty$. This is a contradiction. Hence $t \in T_S(x_0)$. Moreover, $\sup_{y \in N_S(x_0)} \langle y, t \rangle = 0$. So, we have

$$\langle t, \xi \rangle < 0 \text{ for any } \xi \in \partial f(x_0). \tag{3.5}$$

Similarly, from (3.3) there exists $t' \in \mathbb{R}^n$ such that

$$0 > \max_{\xi \in \partial f(x_0)} \langle t', \xi \rangle + \sup_{y \in N_S(x_0)} \langle t', -y \rangle.$$

Thus $-t' \in T_S(x_0)$ and

$$\langle -t', \xi \rangle > 0 \text{ for any } \xi \in \partial f(x_0). \tag{3.6}$$

Now take $x \in S$.

If $f(x) - f(x_0) = 0$, we set $\eta = 0$ for any $\xi \in \partial f(x_0)$, and then $f(x) - f(x_0) = \langle \xi, \eta \rangle$ for any $\xi \in \partial f(x_0)$.

If $f(x) - f(x_0) < 0$, then for any fixed $\xi \in \partial f(x_0)$, we can choose $\alpha > 0$ such that $\langle \alpha t, \xi \rangle = f(x) - f(x_0)$, which is possible by (3.5). Setting $\eta = \alpha t \in T_S(x_0)$, we have $f(x) - f(x_0) = \langle \xi, \eta \rangle$.

If $f(x) - f(x_0) > 0$, then for any fixed $\xi \in \partial f(x_0)$ we can find $\beta > 0$ such that $\langle -\beta t', \xi \rangle = f(x) - f(x_0)$, which is possible by (3.6). Setting $\eta = -\beta t' \in T_S(x_0)$, we have $f(x) - f(x_0) = \langle \eta, \xi \rangle$.

Thus f is infine on S at x_0. Hence (a) holds.

$(a) \Rightarrow (b)$: Assume that f is infine on S at x_0 and that $T_S(x_0)$ is a subspace of $\mathbb{R}^n$. Suppose that there exist $\xi \in \partial f(x_0)$, $y \in N_S(x_0)$ and $y' \in N_S(x_0)$ such that

$$0 = \xi + y - y'. \tag{3.7}$$

By the infineness of f, for any $x \in S$, $f(x) - f(x_0) = \langle \xi, \eta \rangle$. Since $T_S(x_0)$ is a subspace of $\mathbb{R}^n$, for any $x \in S$,

$$f(x) - f(x_0) = \langle \xi, \eta \rangle = \langle y' - y, \eta \rangle = 0.$$

Thus f is constant on S. □

The following proposition is useful to understand condition (b) in Proposition 3.1:

PROPOSITION 3.2. *The following statements are equivalent:*

(1) There exists $\xi \in \partial f(x_0)$ *such that* $0 \in$ *aff*$(\xi + N_S(x_0))$*;*
(2) $0 \in \partial f(x_0) + N_S(x_0) - N_S(x_0)$*; and*
(3) $0 \in$ *co* $\{(\partial f(x_0) + N_S(x_0)) \bigcup (\partial f(x_0) - N_S(x_0))\}$.

Proof. (1) ⇒ (2) : Let (1) hold. Let $r_i \in \mathbb{R}, i = 1, \cdots, m$, and $y_i \in N_S(x_0)$, $i = 1, \cdots, m$, be such that

$$1 = \sum_{i=1}^{m} r_i, 0 = \sum_{i=1}^{m} r_i(\xi + y_i).$$

Let $r_i = r_i' - r_i''$ for $r_i' \geqq 0$ and $r_i'' \geqq 0$. Then we have

$$\begin{aligned} 0 = \xi + \sum_{i=1}^{n} r_i y_i &= \xi + \sum_{i=1}^{n} (r_i' - r_i'') y_i \\ &= \xi + \sum_{i=1}^{n} r_i' y_i - \sum_{i=1}^{n} r_i'' y_i \\ &\in \xi + N_S(x_0) - N_S(x_0). \end{aligned}$$

Thus (2) holds.

(2) $\Rightarrow$ (1): Let $\xi \in \partial f(x_0)$, $y \in N_S(x_0)$ and $y' \in N_S(x_0)$ be such that (3.7) is satisfied. Then we have

$$0 = 2\left(\xi + \frac{y}{2}\right) + (-1)(\xi + y') \in \text{aff}(\xi + N_S(x_0)).$$

So (1) holds.

(2) $\Rightarrow$ (3): Let (2) hold. From (3.7), $0 = \frac{1}{2}(\xi + 2y) + \frac{1}{2}(\xi - 2y')$. So, (3) holds.

(3) $\Rightarrow$ (2): Let (3) hold. Then there exist nonnegative numbers l_i, l_j', and points ξ_i, ξ_j' in $\partial f(x_0)$ such that

$$\begin{aligned} &1 = \sum l_i + \sum l_j' \text{and} \\ &0 = \sum l_i(\xi_i + y_i) + \sum l_j'(\xi_j' - y_j'), \end{aligned}$$

where y_i and y_j' are elements of $N_S(x_0)$. Thus $0 \in \sum l_i \xi_i + \sum l_j' \xi_j' + N_S(x_0) - N_S(x_0)$. Hence (2) holds. □

REMARK 3.1. If $(\partial f(x_0) + N_S(x_0)) \bigcup (\partial f(x_0) - N_S(x_0))$ is convex, then Proposition 3.2 shows that conditions (b) and (c) of Proposition 3.1 are equivalent.

When $S = \mathbb{R}^n$, then $N_S(x_0) = -N_S(x_0) = \{0\}$. So we can obtain the following corollaries.

COROLLARY 3.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a locally Lipschitz function. Then f is infine on $\mathbb{R}^n$ at $x_0 \in \mathbb{R}^n$ if and only if inclusion $0 \in \partial f(x_0)$ implies that f is constant on $\mathbb{R}^n$.*

COROLLARY 3.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be of class C^1. Then f is infine on $\mathbb{R}^n$ at $x_0 \in \mathbb{R}^n$ if and only if condition $f'_{x_0} = 0$ implies that f is constant on $\mathbb{R}^n$.*

Now we give an example involving a nondifferentiable infine function, which shows that the assumption that $T_S(x_0)$ is a subspace of $\mathbb{R}^n$ is essential in Proposition 3.1.

EXAMPLE 3.1. Let $x \in \mathbb{R}$, $x_0 = 0$ and

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \geqq 0 \\ x & \text{if } x < 0 \end{cases}.$$

Then $\partial f(x_0) = [\frac{1}{2}, 1]$. Since $0 \notin \partial f(x_0)$, by Corollary 3.1, f is infine on $S = \mathbb{R}$ at x_0. Now let $S = [0, \infty)$. Then $T_S(x_0) = [0, \infty)$ and $N_S(x_0) = (-\infty, 0]$. Taking for any $x \in S$ and any $\xi \in \partial f(x_0)$, $\eta = [f(x) - f(x_0)]/\xi$, we have $\eta \in T_S(x_0)$ and $f(x) - f(x_0) = \xi \cdot \eta$. Thus f is infine on S at x_0. Notice that for any $\xi \in \partial f(x_0)$, $0 \in \partial f(x_0) + N_S(x_0) - N_S(x_0)$ and

$$0 \in (\partial f(x_0) + N_S(x_0)) \bigcup (\partial f(x_0) - N_S(x_0)).$$

Since f is not constant on S, (b) and (c) of Proposition 3.1 do not hold. Hence the assumption that $T_S(x_0)$ is a subspace of $\mathbb{R}^n$ is essential in the implication $(a) \Rightarrow (b)$ in Proposition 3.1.

EXAMPLE 3.2. Let $S \subset \mathbb{R}^n$ be a convex set. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is of class C^1 and is pseudolinear at $x_0 \in S$ in the sense of [3] (i.e., for any $x \in \mathbb{R}^n$, there exists $p(x) > 0$ such that $f(x) - f(x_0) = p(x) f'_{x_0}(x - x_0)$), then f is infine on S at x_0 with $\eta = p(x)(x - x_0)$ for each $x \in \mathbb{R}^n$.

EXAMPLE 3.3. Let S be any closed subset of $\mathbb{R}^n$. Let $f = g \circ h$, where $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is of class C^1 and is pseudolinear at $h(x_0)$, and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of class C^1 such that $h'_{x_0}(T_S(x_0)) = \mathbb{R}^m$. Then f is infine on S at x_0. Indeed, setting $u = h(x)$, $u_0 = h(x_0)$, we have by pseudolinearity of g that for any $x \in S$

$$\begin{aligned} f(x) - f(x_0) &= g(u) - g(u_0) \\ &= p(u) g'_{u_0}(u - u_0) \\ &= g'_{u_0}(p(u)(u - u_0)), \end{aligned}$$

where $p(u) > 0$ is a suitable number. Since $h'_{x_0}(T_S(x_0)) = \mathbb{R}^m$, there is $\eta \in T_S(x_0)$ such that $p(u)(u - u_0) = h'_{x_0}\eta$. Hence we have

$$f(x) - f(x_0) = g'_{u_0}(h'_{x_0}\eta) = f'_{x_0}\eta.$$

Thus f is infine on S at x_0.

REMARK 3.2. Let $f = g \circ h$, where $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is an affine function and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is of class C^1 such that the Fréchet derivative h'_{x_0} of h at $x_0 \in \mathbb{R}^n$ is surjective. Then g is pseudolinear at $h(x_0)$ with $p \equiv 1$ and hence, as showed in Example 3.3, f is infine on $\mathbb{R}^n$ at x_0. Observe that g being affine is infine on $\mathbb{R}^n$. Thus, if h is bijective then the infineness property of the affine function g is a property invariant to bijective coordinate transformation h. If $g : \mathbb{R}^m \longrightarrow \mathbb{R}$ is convex and differentiable and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a differentiable function such that h'_{x_0} is surjective, then f is invex in the sense of Craven [6, 7]. So, the invexity property of the convex function g is a property invariant to bijective coordinate transformation h. Noticing this fact, Craven [6, 7] introduced the name "*invexity*" which is taken from "*invariant*" and "*convexity*". In a similar way, we used the terminology "*infine*" taken from "*invariant*" and "*affine*".

EXAMPLE 3.4. Assume that $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is locally Lipschitz and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is of class C^1 such that h'_{x_0} has maximal rank, where $x_0 \in \mathbb{R}^n$. Let $f = g \circ h$. If g is infine on $\mathbb{R}^n$ at $h(x_0)$, then f is infine on $\mathbb{R}^n$ at x_0. Indeed, if $0 \in \partial g(h(x_0))$ then by Corollary 3.1 g is constant on $\mathbb{R}^n$. Thus f is constant and hence the infineness of f is obvious. Assume now that $0 \notin \partial g(h(x_0))$. We claim that $0 \notin \partial f(x_0)$ and hence f is infine on $\mathbb{R}^n$ at x_0 (see Corollary 3.1). Indeed, Theorem 2.3.10 of Clarke [4] shows that

$$\partial f(x_0) \subset h'^{\tau}_{x_0} \partial g(h(x_0))$$

where τ denotes the transpose. Hence if $0 \in \partial f(x_0)$, then $h'^{\tau}_{x_0}\xi = 0$ for some $\xi \in \partial g(h(x_0))$. Since h'_{x_0} has maximal rank, this implies that $\xi = 0$, which contradicts the assumption $0 \notin \partial g(h(x_0))$.

Now, let us give two definitions of invex vector-valued functions the first of which is a new one and the second is taken from [6, 7, 19]:

DEFINITION 3.1. Let $f := (f_1, f_2, \cdots, f_m)$ be a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $f_i, i = 1, 2, \cdots, m$, are locally Lipschitz, and S a closed subset of $\mathbb{R}^n$. Then f is said to be invex on S at $x_0 \in S$ if for any $x \in S$ and $\xi_i \in \partial f_i(x_0), \quad i = 1, 2, ..., m$, there exists $\eta \in T_S(x_0)$ such that

$$f_i(x) - f_i(x_0) \geqq \langle \xi_i, \eta \rangle, i = 1, 2, \cdots, m.$$

DEFINITION 3.2. Let $f := (f_1, f_2, \cdots, f_m)$ be a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $f_i, i = 1, 2, \cdots, m$, are locally Lipschitz, and S a closed subset of $\mathbb{R}^n$. Then f is said to be invex on S at $x_0 \in S$ if for any $x \in S$, there exists $\eta \in T_S(x_0)$ such that for any $\xi_i \in \partial f_i(x_0), \quad i = 1, 2, ..., m$,

$$f_i(x) - f_i(x_0) \geqq \langle \xi_i, \eta \rangle, i = 1, 2, \cdots, m,$$

or equivalently,

$$f_i(x) - f_i(x_0) \geqq f_i^0(x_0, \eta), i = 1, 2, \cdots, m.$$

Observe that in practice it is not easy to find the point η required in the definitions of invexity. On the other hand, an explicit formulae of η plays no role in applications of invexity ideas in optimization and duality theories. So, it is interesting to detect invexity properties without knowning η explicitly. We will see in

Proposition 3.3 below that this can be obtained by using Propositions 2.2 and 2.3. Let us introduce the following sets

$$\widehat{E}(x) = \text{cone}\Big[\Big(\bigcup_{i=1}^{m}\big[\partial f_i(x_0)\times\{-\beta_i(x,x_0)\}\big]\Big)\bigcup\big(\partial d_S(x_0)\times\{0\}\big)\Big]$$

$$E(x) = \bigcup_{\xi\in\partial f(x_0)} \text{cl cone}\Big[\Big(\bigcup_{i=1}^{m}\big[\{\xi_i\}\times\{-\beta_i(x,x_0)\}\big]\Big)\bigcup\big(\partial d_S(x_0)\times\{0\}\big)\Big]$$

where $x \in S,\ \xi = (\xi_1, \xi_2, ..., \xi_m) \in \partial f(x_0) := \partial f_1(x_0) \times \partial f_2(x_0) \times \cdots \times \partial f_m(x_0)$ and $\beta_i(x, x_0) = f_i(x) - f_i(x_0),\ i = 1, 2, ..., m.$

PROPOSITION 3.3. *Let $f := (f_1, f_2, \cdots, f_m)$ be a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that $f_i, i = 1, 2, \cdots, m$, are locally Lipschitz, and S a closed subset of $\mathbb{R}^n$.*

Then the following statements are true:

1. f is invex on S at $x_0 \in S$ in the sense of Definition 3.1 if and only if, for any $x \in S,\ (0, 1) \notin E(x)$.
2. f is invex on S at $x_0 \in S$ in the sense of Definition 3.2 if and only if, for any $x \in S,\ (0, 1) \notin cl\ E(x)$.
3. If f is invex on S at $x_0 \in S$ in the sense of Definition 3.2, then f is invex on S at $x_0 \in S$ in the sense of Definition 3.1. The converse holds if for any $x \in S_0$ the set

$$\text{cone}\bigcup_{i=1}^{m}\big[\partial f_i(x_0)\times\{f_i(x_0)-f_i(x)\}\big] + N_S(x_0)\times\{0\} \tag{3.8}$$

is closed, where $S_0 = \{x \in S : f_i(x) - f_i(x_0) < 0 \text{ for some } i\}$.

Proof. 1. Since $\eta \in T_S(x_0) \Leftrightarrow d_S^0(x_0, \eta) \leqq 0$, Definition 3.1 is equivalent to the fact that for any $x \in S$ and $\xi_i \in \partial f_i(x_0),\ i = 1, 2, ..., m$, the system

$$\langle\xi_i, t\rangle \leqq \beta_i(x, x_0),\ i = 1, 2, ..., m,$$
$$d_S^0(x_0, t) \leqq 0$$

is consistent, i.e. by Proposition 2.2

$$(0,1)\notin \text{cl cone}\Big[\Big(\bigcup_{i=1}^{m}\big[\{\xi_i\}\times\{-\beta_i(x,x_0)\}\big]\Big)\bigcup\big(\partial d_s(x_0)\times\{0\}\big)\Big].$$

Since this is true for any $x \in S$ and $\xi_i \in \partial f_i(x_0),\ i = 1, 2, ..., m$, we derive the validity of the first statement of Proposition 3.3.

2. Definition 3.2 is equivalent to the fact that for any $x \in S$ the system

$$f_i^0(x_0, t) \leqq \beta_i(x, x_0),\ i = 1, 2, ..., m,$$
$$d_S^0(x_0, t) \leqq 0$$

is consistent, i.e., by Proposition 2.2 $(0,1) \notin cl\,\widehat{E}(x)$. It remains to observe from Proposition 2.3 that $cl\,\widehat{E}(x) = cl\,E(x)$.

3. The implication

Definition 3.2 $\Rightarrow$ Definition 3.1

is clear from the first two statements of Proposition 3.3. Now let us prove the converse implication under the extra assumption formulated above.

Let $x \in S \setminus S_0$. Then $f_i(x) - f_i(x_0) \geqq 0$ for all i. Thus $\eta = 0$ satisfies Definition 3.2 at this point x. Suppose that f does not satisfy Definition 3.2 at $x \in S_0$. Then for any $\eta \in T_S(x_0)$, $f_i(x) - f_i(x_0) < f_i^0(x_0, \eta)$ for some i. Thus the system

$$\begin{cases} f_i^0(x_0, t) \leqq f_i(x) - f_i(x_0), \quad i = 1, \cdots, m, \\ d_S^0(x_0, t) \leqq 0 \end{cases}$$

of variable $t \in \mathbb{R}^n$ has no solution. Applying Proposition 2.2 and taking (2.1) into account, we have

$$(0,1) \in \mathrm{cl}\,\widehat{E}(x) \subset \mathrm{cl}\left[\mathrm{cone}\left(\bigcup_{i=1}^{m} \partial f_i(x_0) \times \{f_i(x_0) - f_i(x)\}\right) + N_S(x_0) \times \{0\}\right].$$

By assumption (3.8) this yields

$$(0,1) \in \mathrm{cone}\left(\bigcup_{i=1}^{m} \partial f_i(x_0) \times \{f_i(x_0) - f_i(x)\}\right) + N_S(x_0) \times \{0\}\,.$$

Thus $(0,1) \in \mathrm{cone} \bigcup_{i=1}^{m} \big[\{\xi_i\} \times \{f_i(x_0) - f_i(x)\}\big] + N_S(x_0) \times \{0\}$ for some $\xi_i \in \partial f_i(x_0)$, $i = 1, \cdots, m$. This shows that

$$0 = \sum_{i=1}^{m} \lambda_i \xi_i + y,$$

$$1 = \sum_{i=1}^{m} \lambda_i \big(f_i(x_0) - f_i(x)\big) + 0$$

for suitable $\lambda_i \geqq 0$, $i = 1, 2, ..., m$, and $y \in N_S(x_0)$. We claim that there does not exist $\eta \in T_S(x_0)$ such that

$$\langle \xi_i, \eta \rangle \leqq f_i(x) - f_i(x_0),\ i = 1, 2, ..., m.$$

Indeed, otherwise by multiplying both sides of each of these inequalities by λ_i and summing up the obtained inequalities we get

$$\left\langle \sum_{i=1}^{m} \lambda_i \xi_i, \eta \right\rangle \leqq -\sum_{i=1}^{m} \lambda_i \big[f_i(x_0) - f_i(x)\big]$$

or, equivalently, $\langle -y, \eta \rangle \leqq -1$. This contradicts inequality $\langle -y, \eta \rangle \geqq 0$ which is valid since $\eta \in T_S(x_0)$ and $y \in N_S(x_0)$. Thus, f is not invex on S at x_0 in the sense of Definition 3.1. □

The following result is a direct consequence of Proposition 3.3.

COROLLARY 3.3. *Let $f := (f_1, f_2, ..., f_m)$ be a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that f_i, $i = 1, 2, ..., m$, are of class C^1, and S a closed subset of $\mathbb{R}^n$. Then Definitions 3.1 and 3.2 coincide, and the invexity of f on S at x_0 is characterized by the condition that, for any $x \in S$,*

$$(0, 1) \notin \text{ cl cone} \left[\left(\bigcup_{i=1}^{m} \left[\{ f'_{ix_0} \} \times \{ f_i(x_0) - f_i(x) \} \right] \right) \bigcup \left(\partial d_S(x_0) \times \{0\} \right) \right].$$

Let us observe from Proposition 3.3 that the difference between Definitions 3.1 and 3.2 is small in the sense that for any $x \in S$ Definition 3.1 requires that the point (0, 1) does not belong to $E(x)$ while Definition 3.2 requires that this point does not belong to the closure of the same set $E(x)$. Observe also that if for any $x \in S$ $\widehat{E}(x)$ is closed then the two definitions of invexity are equivalent since by Proposition 2.3 *cl* $E(x) = E(x)$. We now show that in this case invexity can be characterized by the following condition (!):

$$\left[\lambda_i \geqq 0,\ i = 1, 2, ..., m;\ 0 \in \sum_{i=1}^{m} \lambda_i \partial f_i(x_0) + \partial d_S(x_0) \right]$$
$$\Rightarrow \left[\sum_{i=1}^{m} \lambda_i (f_i(x) - f_i(x_0)) \geqq 0 \quad \forall x \in S \right].$$

Indeed, since $\widehat{E}(x)$ is closed we have *cl* $E(x) = E(x) = \widehat{E}(x)$ (see Proposition 2.3). Therefore, by Proposition 3.3 each notion of invexity is equivalent to the condition that $(0, 1) \notin \widehat{E}(x)$ for any $x \in S$. But the last condition is equivalent to condition (!) and hence, our desired conclusion follows.

If invexity is understood in the global sense (i.e., if $S = X$) then $\partial d_S(x_0) = \{0\}$. In this special case, condition (!) is equivalent to a known characterization of invexity of Craven [11, Theorem 4]. In other words, our characterization of invexity expressed by condition (!) is an extension of that of Craven [11, Theorem 4] to the case where $S \neq \mathbb{R}^n$.

COROLLARY 3.4. *If $m = 1$, that is f is real-valued, then the two definitions of invexity are equivalent.*

Proof. Let $S_0 = \{x \in S : f(x) - f(x_0) < 0\}$ and $x \in S_0$. To apply Proposition 3.3 we must show the closedness of the set

$$\text{cone}\left[\partial f(x_0) \times \{ f(x_0) - f(x) \} \right] + N_S(x_0) \times \{0\}. \tag{3.8$'$}$$

Indeed, let (p_j, q_j) be a sequence of elements of set $(3.8)'$ such that

$$\lim_{j\to\infty}(p_j, q_j) = (p, q).$$

We shall prove that (p, q) is also an element of set $(3.8)'$. Indeed, let $\lambda_j \geqq 0$, $\xi_j \in \partial f(x_0)$ and $y_j \in N_S(x_0)$ be such that $\lambda_j\xi_j + y_j = p_j$ and $\lambda_j\big(f(x_0) - f(x)\big) = q_j + 0$. Since $q_j \to q$ and $f(x_0) - f(x) \neq 0$ we derive from the last equality that $\lambda_j \to \lambda_0 = q/[f(x_0) - f(x)]$. On the other hand, because of the compactness of $\partial f(x_0)$ we may assume, by taking a subsequence if necessary, that ξ_j converges to some point $\xi \in \partial f(x_0)$. Hence $y_j = p_j - \lambda_j\xi_j$ converges to $p - \lambda_0\xi \in N_S(x_0)$ by the closedness of $N_S(x_0)$. Thus, $p = \lambda_0\xi + y$ and $q = \lambda_0\big(f(x_0) - f(x)\big) + 0$, with suitable $y \in N_S(x_0)$. This proves that (p, q) is also an element of set $(3.8)'$. □

Let us observe from the above discussion that the difference between Definitions 3.1 and 3.2 is small, and that by Corollary 3.4 they are equivalent in case $m = 1$. It is then natural to ask if this equivalence holds for $n \geqq 2$. The following example, due to Tuan [26], gives the negative answer to this question.

EXAMPLE 3.5. Let $n = m = 2$ and $S = \mathbb{R}^2$. Let $x = (y, z) \in \mathbb{R}^2$, $x_0 = (0, 0) \in \mathbb{R}^2$ and let $f = (f_1, f_2)$ where $f_2(x) = f_2(y, z) = (y^2 + z^2)^{1/2} + z$ and f_1 is any function of class C^1 satisfying the following conditions: $f_1(0, 0) = 0$, $f_1(0, -1) = -1$ and $f'_{1x_0} = (1, 0)$. As examples of f_1 we can take $f_1(x) = f_1(y, z) = y + z^{2p+1}$ or $f_1(x) = f_1(y, z) = y - z^{2p}$ where $p \geqq 1$ is a fixed integer.

We have

$$\partial f_1(x_0) = \{(1, 0)\},$$
$$\partial f_2(x_0) = \{(u, v) \in \mathbb{R}^2 : u^2 + (v - 1)^2 \leqq 1\}.$$

We claim that f is invex on $S = \mathbb{R}^n$ at x_0 in the sense of Definition 3.1. Indeed, let $(y, z) \in \mathbb{R}^n$ and $(u, v) \in \partial f_2(x_0)$. Observe that $f_2(y, z) \geqq |z| + z \geqq 0$ and $v \geqq 0$, and that $v = 0 \Rightarrow u = 0$. Taking account of these observations we see that the point $\eta = (\eta_1, \eta_2)$ where

$$\eta_1 = f_1(y, z),$$
$$\eta_2 = \begin{cases} 0 & \text{if } v = 0, \\ v^{-1}[f_2(y, z) - f_1(y, z)u] & \text{if } v > 0, \end{cases}.$$

satisfies the condition required in Definition 3.1.

Now, let us prove that f is not invex on $S = R^n$ at x_0 in the sense of Definition 3.2. Indeed, to prove this claim it suffices to take $x = (0, -1) \in \mathbb{R}^2$ and show that the system

$$f_1(x) - f_1(x_0) \geqq f_1^0(x_0, \eta),$$
$$f_2(x) - f_2(x_0) \geqq f_2^0(x_0, \eta)$$

of variable $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ has no solution, where $f_i^0(x_0, \eta) = \max\{u\eta_1 + v\eta_2 : (u, v) \in \partial f_i(x_0)\}$ (see (2.1)). Under the assumptions of Example 3.5 this system is of the form

$$\begin{aligned} -1 &\geqq \eta_1, \\ 0 &\geqq \max\{u\eta_1 + v\eta_2 : u^2 + (v-1)^2 \leqq 1\} \end{aligned}$$

and has no solution. This can be seen from the remark that for all $\eta_1 \neq 0$ the right side of the last inequality is always positive.

Now we define pseudoinvex vector-valued function.

DEFINITION 3.3. *Let $f := (f_1, f_2, \cdots, f_m)$ be a vector-valued function such that $f_i, i = 1, 2, \cdots, m$, are locally Lipschitz, and S a closed subset of $\mathbb{R}^n$. Then f is said to be pseudoinvex on S at $x_0 \in S$ if*

$$\forall x \in S \quad \forall \xi_i \in \partial f_i(x_0),\ i = 1, 2, \ldots, m,\ \exists \eta \in T_S(x_0) \quad \forall i$$

$$\big[\langle \xi_i, \eta \rangle \geqq 0 \Rightarrow f_i(x) \geqq f_i(x_0)\big].$$

Roughly speaking, f is pseudoinvex on S at $x_0 \in S$ if each component f_i is pseudoinvex on S at $x_0 \in S$ with the same η for each component.

REMARK 3.3. If $f := (f_1, f_2, \cdots, f_m)$ is invex on S at $x_0 \in S$ in the sense of Definition 3.1 or Definition 3.2, then f is pseudoinvex on S at $x_0 \in S$.

REMARK 3.4. By using Corollary 3.4 and a result of Phuong, Sach and Yen [19, Theorem 4.1] we can prove that the pseudoinvexity of a real-valued function f on S at x_0 is equivalent to the invexity of f on S at x_0 in the sense of Definition 3.1 or Definition 3.2. Thus, there is no distinction between pseudoinvex and invex functions. This was shown in [1] for the case when f is a differentiable function and S coincides with the whole space $\mathbb{R}^n$.

From Remark 3.4 we see that everywhere in the formulation of the following Proposition 3.4 the term "pseudoinvex" can be replaced by "invex".

PROPOSITION 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and S a closed subset of $\mathbb{R}^n$. Consider the following statements:*

(a) f is infine on S at $x_0 \in S$;

(d) $-f$ is pseudoinvex on S at x_0 and, for any $x \in S$, there exists $t \in T_S(x_0)$ such that

$$f(x) - f(x_0) = f^0(x_0, t); \tag{3.9}$$

(e) f is pseudoinvex on S at x_0 and for any $x \in S$ there exists $t \in T_S(x_0)$ such that

$$\big[-f(x)\big] - \big[-f(x_0)\big] = (-f)^0(x_0, t).$$

Then (d) $\Rightarrow$ (a), and (e) $\Rightarrow$ (a).

Proof. (d) $\Rightarrow$ (a): Let $x \in S$ and $\xi \in \partial f(x_0)$. Since $-f$ is pseudoinvex on S at x_0 and $-\xi \in \partial(-f)(x_0)$, there exists $t' \in T_S(x_0)$ such that

$$\langle -\xi, t' \rangle \geqq 0 \Rightarrow f(x) - f(x_0) \leqq 0. \tag{3.10}$$

By (3.9), $f(x) - f(x_0) \geqq \langle \xi, t \rangle$. If $f(x) - f(x_0) = \langle \xi, t \rangle$, taking $\eta = t$, we have

$$f(x) - f(x_0) = \langle \xi, \eta \rangle.$$

Assume that $f(x) - f(x_0) > \langle \xi, t \rangle$. If $f(x) - f(x_0) < 0$, then $\langle \xi, t \rangle < 0$. We can choose $\alpha > 0$ such that $f(x) - f(x_0) = \alpha\langle \xi, t \rangle$, and hence letting $\eta = \alpha t$, we have

$$\eta \in T_S(x_0) \text{and} f(x) - f(x_0) = \langle \xi, \eta \rangle.$$

If $f(x) - f(x_0) > 0$, by (3.10) $\langle \xi, t' \rangle > 0$. We can take $\beta > 0$ such that $f(x) - f(x_0) = \beta\langle \xi, t' \rangle$, and hence letting $\eta = \beta t'$, we have

$$\eta \in T_S(x_0) \text{and} f(x) - f(x_0) = \langle \xi, \eta \rangle.$$

If $f(x) - f(x_0) = 0$, letting $\eta = 0$, we have $\eta \in T_S(x_0)$ and $f(x) - f(x_0) = \langle \xi, \eta \rangle$. Consequently, f is infine on S at x_0.

(e) $\Rightarrow$ (a): Applying implication (d) $\Rightarrow$ (a) with $-f$ instead of f, we see that condition (e) implies the infineness of $-f$ (and hence the infineness of f) on S at x_0. □

REMARK 3.5. The real-valued locally Lipschitz function f satisfying (3.9) in Proposition 3.4 may not be infine. For example, $f(x) = |x|$ and $x_0 = 0$.

To give a sufficient condition for the infineness property of f by means of the pseudoinvexity of $\overline{f} := (f, -f)$ we first establish an elementary result.

LEMMA 3.1. *Let a, b, c* $\in \mathbb{R}$ *be such that*

$$a + b \geqq 0, \tag{3.11}$$
$$a \geqq 0 \Rightarrow c \geqq 0, \tag{3.12}$$
$$b \geqq 0 \Rightarrow c \leqq 0. \tag{3.13}$$

Then

$$\exists \lambda \geqq 0 \text{ such that } c = \lambda a \tag{3.14}$$

and

$$\exists \mu \leqq 0 \text{ such that } c = \mu b. \tag{3.15}$$

Proof. Let us prove (3.14).

If $a = 0$, it follows from (3.12) and (3.13) that $c = 0$. So, (3.14) holds.

If $a > 0$, then by (3.12) $c \geqq 0$. So, (3.14) holds.

If $a < 0$, then by (3.11) $b \geqq 0$ and hence, by (3.13), $c \leqq 0$. So, (3.14) holds.

To prove (3.15) let us set $c' = -c$ and change the role of a and b. Then we have from (3.14) that $c' = \lambda' b$ $(\lambda' \geqq 0)$. Thus $-c = \lambda' b$ and hence, $c = \mu b$with $\mu = -\lambda' \leqq 0$. □

REMARK 3.6. *In general, we cannot claim that $\lambda > 0$ and $\mu > 0$. For example,* $a = 1, b = 1, c = 0$.

PROPOSITION 3.5. *Consider the following statements:*

(a) f is infine on S at x_0.

(g) $\bar{f} := (f, -f)$ is pseudoinvex on S at x_0.

Then (g) $\Rightarrow$ (a).

Proof. Let $x \in S$ and $\xi \in \partial f(x_0)$. In view of (2.2) $-\xi \in \partial(-f)(x_0)$. By pseudoinvexity of $\bar{f}$, there exists $t \in T_S(x_0)$ such that

$$\langle \xi, t \rangle \geqq 0 \Rightarrow f(x) - f(x_0) \geqq 0,$$

and

$$\langle -\xi, t \rangle \geqq 0 \Rightarrow f(x) - f(x_0) \leqq 0.$$

Applying Lemma 3.1 with $a = \langle \xi, t \rangle$, $b = \langle -\xi, t \rangle$ and $c = f(x) - f(x_0)$, we find $\lambda \geqq 0$ such that

$$f(x) - f(x_0) = \lambda \langle \xi, t \rangle = \langle \xi, \lambda t \rangle = \langle \xi, \eta \rangle \text{ with } \eta = \lambda t \in T_S(x_0). \qquad \square$$

REMARK 3.7. If $\partial f(x_0)$ is not a singleton, then f may be infine on $\mathbb{R}^n$ at x_0 while $\bar{f} := (f, -f)$ may not be invex on $\mathbb{R}^n$ at x_0 in the sense of Definition 3.1. Indeed, consider again Example 3.1. We have seen that f is infine on $S = \mathbb{R}$ at $x_0 = 0$. Now let $x > 0$. Take $\bar{\xi} = 1/2, \xi = 1$. Then $\xi \in \partial f(x_0)$ and $-\bar{\xi} \in \partial(-f)(x_0)$. Assume that there exists $\eta \in \mathbb{R}^n$ such that

$$\begin{aligned} 1/2x &= f(x) - f(x_0) \geqq \langle \xi, \eta \rangle = \eta, \\ -1/2x &= [-f(x)] - [-f(x_0)] \geqq \langle -\bar{\xi}, \eta \rangle = -1/2\eta. \end{aligned}$$

From the just written results it follows that $2\eta \leqq x \leqq \eta$. This is a contradiction since $x > 0$. Thus $\bar{f} = (f, -f)$ is not invex on $\mathbb{R}$ at x_0. (Observe from the diagram given at the end of this section that $\bar{f}$ is pseudoinvex on $\mathbb{R}^n$ at x_0.)

The following propositions are useful for connecting Propositions 3.1, 3.4 and 3.5:

PROPOSITION 3.6. *Condition (c) of Proposition 3.1 implies both conditions (d) and (e) of Proposition 3.4.*

Proof. (c) $\Rightarrow$ (d): It suffices to consider the case when

$$0 \notin (\partial f(x_0) + N_S(x_0)) \cup (\partial f(x_0) - N_S(x_0)). \tag{3.16}$$

Indeed, otherwise, by condition (c), f is constant on S and hence condition (d) trivially holds.

Obviously, (3.16) yields (3.2) and (3.3). The argument used in the proof of Proposition 3.1 shows that there exist $t \in T_S(x_0)$ and $-t' \in T_S(x_0)$ such that (3.5) and (3.6) hold. Hence $f^0(x_0, t) < 0$ and $f^0(x_0, -t') > 0$. Thus, for any $x \in S$ there exists $a > 0$ such that $f(x) - f(x_0) = af^0(x_0; t'') = f^0(x_0, at'')$ where

$$t'' = \begin{cases} 0 & \text{if } f(x) - f(x_0) = 0, \\ t & \text{if } f(x) - f(x_0) < 0, \\ -t' & \text{if } f(x) - f(x_0) > 0. \end{cases}$$

Setting $\eta = at''$ we obtain $\eta \in T_S(x_0)$ and $f(x) - f(x_0) = f^0(x_0, \eta)$ (i.e., (3.9) holds with η instead of t).

Now we prove the pseudoinvexity of $-f$. For all x of S and $\xi \in \partial(-f)(x_0)$, take $\eta' = -t'$. We have $\langle \xi, \eta' \rangle < 0$ (see (3.6) and observe that $-\xi \in \partial f(x_0)$). So nothing is required for the sign of $(-f)(x) - (-f)(x_0)$ and the pseudoinvexity of $-f$ is established.

(c) $\Rightarrow$ (e): Let us observe that condition (c) is equivalent to the following condition

$(c)'$ If $0 \in (\partial(-f)(x_0) + N_S(x_0)) \bigcup (\partial(-f)(x_0) - N_S(x_0))$ then $-f$ is constant on S.

Hence applying implication (c) $\Rightarrow$ (d) with $-f$ instead of f, we obtain condition (e). □

PROPOSITION 3.7. *Condition (c) of Proposition 3.1 implies condition (g) of Proposition 3.5.*

Proof. It is easy to see that condition (g) is equivalent to the fact that for any $x \in S$, $\xi \in \partial f(x_0)$ and $\bar{\xi} \in \partial(-f)(x_0)$, there is $\eta \in T_S(x_0)$ such that

$$f(x) - f(x_0) < 0 \ \Rightarrow \ \langle \xi, \eta \rangle < 0, \tag{3.17}$$

$$f(x) - f(x_0) > 0 \ \Rightarrow \ \langle \bar{\xi}, \eta \rangle < 0. \tag{3.18}$$

Now if $0 \in (\partial f(x_0) + N_S(x_0)) \cup (\partial f(x_0) - N_S(x_0))$, then by (c), f is constant on S and hence, (g) holds. If (3.16) holds, then, as we showed in the proof of Proposition 3.1, there exist $t \in T_S(x_0)$ and $-t' \in T_S(x_0)$ satisfying (3.5) and (3.6). Hence, $f^0(x_0, t) < 0$ and $(-f)^0(x_0, -t') < 0$. From this it follows that (3.17) and (3.18) hold if for any $x \in S$, $\xi \in \partial f(x_0)$ and $\bar{\xi} \in \partial(-f)(x_0)$ we set

$$\eta = \begin{cases} -t' & \text{if} \quad f(x) - f(x_0) > 0, \\ t & \text{if} \quad f(x) - f(x_0) < 0. \end{cases}$$

□

On the basis of the results of this section we can give a diagram for connecting conditions in Propositions 3.1, 3.4 and 3.5.

THEOREM 3.1. *The following diagram is true:*

$$\begin{array}{ll} & \Rightarrow (d) \Rightarrow \\ (a) \text{“} \Rightarrow \text{”} (b) \Rightarrow (c) & \Rightarrow (e) \Rightarrow \quad (a). \\ & \Rightarrow (g) \Rightarrow \end{array}$$

Here “⇒” means implication under the condition that $T_S(x_0)$ is a subspace of $\mathbb{R}^n$.

Thus, under the last assumption all statements (a), (b), (c), (d), (e) and (g) are equivalent.

4. Nonsmooth alternative theorems and application to a vector optimization problem

Let S be a nonempty closed subset of $\mathbb{R}^n$, and $J = \{1, 2, ..., p\}$ and $K = \{1, 2, ..., l\}$ be index sets. Let $g = (g_1, g_2, ..., g_p)$ and $h = (h_1, h_2, ..., h_l)$ be vector-valued functions with locally Lipschitz components defined on $\mathbb{R}^n$. We say that $(g; h)$ is *invex-infine* on S at $x_0 \in S$ if for any $x \in S$, $\bar{\xi}_j \in \partial g_j(x_0)$ $(j \in J)$ and $\bar{\bar{\xi}}_k \in \partial h_k(x_0)$ $(k \in K)$ there exists $\eta \in T_S(x_0)$ such that

$$g_j(x) - g_j(x_0) \geqq \langle \bar{\xi}_j, \eta \rangle \qquad (j \in J),$$

$$h_k(x) - h_k(x_0) = \langle \bar{\bar{\xi}}_k, \eta \rangle \qquad (k \in K).$$

If h is absent then this definition is exactly Definition 3.1 of invexity of g. If g is absent then this definition reduces to that of infininess of each component h_k of h, with an additional requirement that, for fixed $x \in S$ and $\bar{\bar{\xi}}_k \in \partial h_k(x_0)$ $(k \in K)$, the point $\eta \in T_S(x_0)$ appearing in the definition of infineness must be the same for all components h_k. Roughly speaking, $(g; h)$ is invex-infine if the first part of this vector-valued map (i.e., map g) is invex and the second part (i.e., map h) is infine, with the same η.

Let us introduce the function

$$q(\cdot) = \max \left\{ \max_{j \in J} g_j(\cdot), \ \max_{k \in K} |h_k(\cdot)| \right\}$$

and, for $x_0 \in S$, let us set

$$J_0 = J(x_0) = \{j \in J : g_j(x_0) = q(x_0)\},$$

$$K_0 = K(x_0) = \{k \in K : |h_k(x_0)| = q(x_0)\}.$$

Observe that one of the last two index sets may be empty, but their union must be nonempty. We will denote by α_{J_0} the vector with components α_j $(j \in J_0)$. Similarly, g_{J_0} is used to denote the vector-valued function with components g_j $(j \in J_0)$.

THEOREM 4.1. *Assume that q attains its minimum on S at a point $x_0 \in S$ and $(g_{J_0}; h_{K_0})$ is invex-infine on S at x_0. Then either*

(a) System

$$g(x) \leqq 0, \quad h(x) = 0, \quad x \in S \tag{4.1}$$

has a solution or

(b) There are vectors $\lambda_J \geqq 0$, μ_K and a real number $\epsilon > 0$ such that for any $x \in S$

$$\sum_{j \in J} \lambda_j g_j(x) + \sum_{k \in K} \mu_k h_k(x) > \epsilon \tag{4.2}$$

but never both.

Proof. Obviously, statements (a) and (b) can not be satisfied simultanously. It remains to prove that statement (b) holds if system (4.1) is inconsistent. Indeed, under the last assumption $q(x) > 0$ for all $x \in S$. Since q attains its minimum at $x_0 \in S$, there exists $\epsilon > 0$ such that

$$q(x_0) > \epsilon. \tag{4.3}$$

By the optimality of x_0, we have from Clarke [4, p. 52, Corollary] that

$$0 \in \partial q(x_0) + N_S(x_0), \tag{4.4}$$

where $N_S(x_0)$ is defined by (2.4).

Setting

$$J' = J \cup K, \quad J'_0 = J_0 \cup K_0 \tag{4.5}$$

and

$$g_k(x) = |h_k(x)| \qquad \text{for } k \in K, \tag{4.6}$$

we see that q is the maximum of finitely many functions

$$q(x) = \max_{j \in J'} g_j(x),$$

and hence, we easily check that

$$q^0(x_0, x) \leqq \max_{j \in J'_0} g_j^0(x_0, x). \tag{4.7}$$

Observe from (4.4) and (2.4) that there exists $v \in \partial q(x_0)$ such that $\langle -v, \xi \rangle \leqq 0$ for any $\xi \in T_S(x_0)$. Thus, taking (2.1) into account we get

$$q^0(x_0, \xi) \geqq 0 \qquad \text{for any} \quad \xi \in T_S(x_0). \tag{4.8}$$

From (4.7) and (4.8) we derive the inconsistency of the following system of convex inequalities of variable $\xi \in \mathbb{R}^n$

$$g_j^0(x_0, \xi) < 0 \qquad (j \in J_0'),$$
$$d_S^0(x_0, \xi) \leqq 0.$$

By Proposition 2.1

$$0 \in \ co \left\{ \bigcup_{j \in J_0'} \partial g_j(x_0) \right\} + \text{cl cone } \partial d_S(x_0) \tag{4.9}$$

or equivalently, by (2.5)

$$0 \in \ \text{co} \left\{ \bigcup_{j \in J_0'} \partial g_j(x_0) \right\} + N_S(x_0). \tag{4.10}$$

Thus, there exist $v_j \in \partial g_j(x_0)$ and $\lambda_j \geqq 0$ $(j \in J_0')$ such that

$$\sum_{j \in J_0'} \lambda_j = 1, \tag{4.11}$$

$$-\sum_{j \in J_0'} \lambda_j v_j \in N_S(x_0). \tag{4.12}$$

Observe that, for $k \in K_0$, $g_k(x_0) = |h_k(x_0)| > 0$ (see (4.6) and (4.3)). Hence, by Clarke [4, p.42, Theorem 2.3.9] $\partial g_k(x_0) = a_k \partial h_k(x_0)$ where $a_k := \operatorname{sign} h_k(x_0)$. Therefore, since $J_0' = J_0 \cup K_0$ we can rewrite (4.12) as

$$-\sum_{j \in J_0} \lambda_j v_j - \sum_{k \in K_0} \mu_k v_k' \in N_S(x_0), \tag{4.13}$$

where $\mu_k = \lambda_k a_k$ and $v_k' = a_k v_k \in \partial h_k(x_0)$ for any $k \in K_0$. (Observe that $a_k := \operatorname{sign} h_k(x_0)$.) By the invex-infineness of $(g_{J_0}; h_{K_0})$, for any $x \in S$ there exists $\eta(x) \in T_S(x_0)$ such that

$$g_j(x) - g_j(x_0) \geqq \langle v_j, \eta(x) \rangle \qquad (j \in J_0), \tag{4.14}$$
$$h_k(x) - h_k(x_0) = \langle v_k', \eta(x) \rangle \qquad (k \in K_0). \tag{4.15}$$

Multiplying both sides of (4.14) by λ_j and both sides of (4.15) by μ_k, and summing up the obtained inequalities and equalities we get, for any $x \in S$,

$$\sum_{j \in J_0} \lambda_j \big[g_j(x) - g_j(x_0) \big] + \sum_{k \in K_0} \mu_k \big[h_k(x) - h_k(x_0) \big] \geqq$$
$$\Big\langle \sum_{j \in J_0} \lambda_j v_j + \sum_{k \in K_0} \mu_k v_k', \ \eta(x) \Big\rangle. \tag{4.16}$$

Making use of (4.13) and observing that $\eta(x) \in T_S(x_0)$ we conclude that the right side of (4.16) is nonnegative. Therefore, for any $x \in S$

$$\begin{aligned}\sum_{j\in J_0}\lambda_j g_j(x) + \sum_{k\in K_0}\mu_k h_k(x) &\geqq \sum_{j\in J_0}\lambda_j q(x_0) + \sum_{k\in K_0}\lambda_k q(x_0)\\ &= q(x_0) \qquad \text{(see (4.11))}\\ &> \epsilon \qquad \text{(see (4.3))}.\end{aligned}$$

(Observe that $g_j(x_0) = \left|h_k(x_0)\right| = q(x_0)$ for all $j \in J_0$ and $k \in K_0$.) Thus (4.2) holds, where we set $\lambda_j = \mu_k = 0$ for $j \notin J_0$ and $k \notin K_0$. □

Observe that the inconsistency of system (4.1) is equivalent to the inequality $\min_{x\in S} q(x) > 0$. Thus, by means of function q the inconsistency of a system of inequalities and equalities reduces to a simple minimization problem of a real-valued function on a subset S. Therefore, a necessary optimality condition of this problem can be used as a starting point to derive a necessary condition for the inconsistency of system (4.1). (In fact, we have seen from the above proof that a combination of this necessary optimality condition and the invex-infineness property yields the desired statement (b) of Theorem 4.1.) This is a starting idea to introduce the function q in Theorem 4.1. The construction of q is also motivated by an approach of Clarke [5] where a function similar to q is used to prove the existence of Lagrange multiliers in an optimization problem involving inequalities and equalities. When h is absent, q is exactly the function defined in [2] to establish an invex Gordan Theorem. However, the approach of [2] is quite different from that of this paper.

Observe also that

$$\text{statement (b) of Theorem 4.1} \Leftrightarrow \min_{x\in S} q(x) > 0.$$

Indeed, the implication '⇒' is obvious, and the converse implication is obtained from the above proof of Theorem 4.1. Thus, the function q plays an intermediate role in proving the following equivalence:

statement (b) of Theorem 4.1 ⇔ inconsistency of system (4.1).

Applying Theorem 4.1, we can obtain the following corollary which is very closely related to Bohnenblust–Karlin–Shapley Theorem found in [17, p. 67].

COROLLARY 4.1. *Let S be a nonempty compact subset of $\mathbb{R}^n$, $\{g_j\}_{j\in J'}$ and $\{h_k\}_{k\in K'}$ be (finite or infinite) families of locally Lipschitz functions such that for any $x_0 \in S$ and finite subsets J of J' and K of K', the vector-valued function $(g_J; h_K)$ is invex-infine on S at x_0. Then either*

(a) System

$$g_j(x) \leqq 0 \ (j \in J'), h_k(x) = 0 \ (k \in K'), x \in S \tag{4.17}$$

has a solution
or

(b) There are finite subsets J of J' and K of K', real numbers $\epsilon > 0$, $\lambda_j \geqq 0$ $(j \in J)$, $\mu_k \in \mathbb{R}$ $(k \in K)$ such that, for any $x \in S$,

$$\sum_{j\in J} \lambda_j g_j(x) + \sum_{k\in K} \mu_k h_k(x) > \epsilon, \tag{4.18}$$

but never both.

Proof. Obviously, (a) and (b) cannot be satisfied simultaneously. Assume now that (a) does not hold, then as in [17, p. 68], there are finite subsets $J \subset J'$ and $K \subset K'$ and $\epsilon_j > 0$ $(j \in J)$ such that system

$$g_j(x) - \epsilon_j \leqq 0 \ (j \in J), h_k(x) = 0 \ (k \in K), \quad x \in S \tag{4.19}$$

has no solution. Since S is compact, applying Theorem 4.1 yields (4.18). □

REMARK 4.1. If g_j are convex functions on the whole space, h_k are linear functions and S is a nonempty compact convex set, then all assumptions of Corollary 4.1 are satisfied. Hence under these hypotheses, if (4.17) has no solution, then (4.18) holds. This conclusion is stronger than the corresponding conclusion of the Bohnenblust–Karlin–Shapley Theorem in [17] which says that the inconsistency of (4.17) implies the existence of J, K, λ_j and μ_k such that for any $x \in S$ the left side of (4.18) is only nonnegative. However, we have to use assumptions stronger than those in [17]. Namely, we must assume that all functions g_j are convex on the whole space $\mathbb{R}^n$ while in [17] g_j are assumed to be convex and lower semicontinuous on S only.

Before going further let us recall some notions of [21]. Let S be a nonempty convex subset of $\mathbb{R}^n$ and denote the set $\{y \in \mathbb{R}^n : S + y \subset S\}$ by 0^+S. Then each direction $y \neq 0$ in 0^+S is called [21] a direction of recession of S.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\psi 0^+$ a function whose epigraph equals $0^+(\operatorname{epi}\psi)$, where $\operatorname{epi}\psi$ is the epigraph of the function ψ. Then the set $\left\{y \in \mathbb{R}^n : (\psi 0^+)(y) \leqq 0\right\}$ is called in [21] the recession cone of ψ, and each direction $y \neq 0$ in this set is called [21] a direction of recession of ψ.

Using Theorem 4.1, we can obtain the following corollary, which is a special case of Theorem 21.3 in [21] and tell us the relationship between Theorem 4.1 and Theorem 21.3 in [21].

COROLLARY 4.2. *Let J be a finite index set, g_j $(j \in J)$ a convex function from $\mathbb{R}^n$ to $\mathbb{R}$ and S a nonempty closed convex subset of $\mathbb{R}^n$. Assume that the functions g_j have no common direction of recession which is also a direction of recession of S. Then either*

(a) System

$$g_j(x) \leqq 0 \ (j \in J), \quad x \in S$$

has a solution
or

(b) There are $\lambda_j \geqq 0$ $(j \in J)$ *and* $\epsilon > 0$ *such that for any* $x \in S$

$$\sum_{j \in J} \lambda_j g_j(x) > \epsilon,$$

but never both.

Proof. The functions g_j are invex on S at every point $x_0 \in S$ since g_j are real-valued convex function and S is a convex set. Let $q(x) = \max \left\{ g_j(x) : j \in J \right\}$ for any $x \in \mathbb{R}^n$. Then by Theorem 9.4 in [21], the function q and the set S have no common direction of recession. So, by Theorem 27.3 in [21], the function q attains its minimum on S. Hence, it follows from Theorem 4.1 that the conclusion of Corollary 4.2 holds. □

Now we will give a Gordan type alternative theorem for invex-infine functions.

Let $I = \{1, 2, \cdots, m\}$, $J = \{1, 2, \cdots, p\}$ and $K = \{1, 2, \cdots, l\}$ be index sets, and let $f = (f_1, f_2, ..., f_m)$, $g = (g_1, g_2, ..., g_p)$ and $h = (h_1, h_2, ..., h_l)$ be vector-valued maps with locally Lipschitz components defined on $\mathbb{R}^n$. Let S be a nonempty closed subset of $\mathbb{R}^n$. Let us set

$$s(\cdot) = \max \left\{ \max_{i \in I} f_i(\cdot), \max_{i \in J} g_j(\cdot), \max_{k \in K} |h_k(\cdot)| \right\}.$$

For $x_0 \in S$, let us set

$$I_0 := I(x_0) = \{i \in I : f_i(x_0) = s(x_0)\},$$
$$J_0 := J(x_0) = \left\{ j \in J : g_j(x_0) = s(x_0) \right\},$$
$$K_0 := K(x_0) = \{k \in K : |h_k(x_0)| = s(x_0)\},$$
$$J_0' = J_0 \cup K_0.$$

DEFINITION 4.1. We say that condition (CQ) is satisfied at x_0 if there do not exist real numbers β_j $(j \in J_0')$, not all zero, such that $\beta_j \geqq 0$ $(j \in J_0)$ and

$$0 \in \sum_{j \in J_0'} \beta_j \partial g_j(x_0) + N_S(x_0)$$

where we set $g_j = h_j$ for $j \in K_0$.

Let us observe that if $J_0 = \emptyset$ then condition (CQ) means that the validity of inclusion

$$0 \in \sum_{k \in K_0} \beta_k \partial h_k(x_0) + N_S(x_0)$$

implies that $\beta_k = 0$ $(\forall k \in K_0)$. This becomes the requirement of the linear independence of the Fréchet derivatives h'_{kx_0} if we assume additionally that $S = R^n$ and h_k $(k \in K_0)$ are of class C^1.

If $K_0 = \emptyset$ then condition (CQ) means that

$$0 \notin co\left\{\bigcup_{j\in J_0} \partial g_j(x_0)\right\} + N_S(x_0).$$

This becomes condition (CQ) used in [24] if S is an open set (which implies that $N_S(x_0) = \{0\}$).

We will say that $(f, g; h)$ is *invex-infine* on S at x_0 if $(\bar{g}; h)$ is invex-infine on S at x_0 where $\bar{g} = (f, g)$ is the vector-valued map with components $f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_p$.

THEOREM 4.2. *Assume that the function s attains its minimum on S at $x_0 \in S$, $(f_{I_0}, g_{J_0};\ h_{K_0})$ is invex-infine on S at x_0, and condition (CQ) is satisfied at x_0. Assume, in addition, that system*

$$f(x) \leqq 0, \quad g(x) \leqq 0, \quad h(x) = 0, x \in S \tag{4.20}$$

has at least a solution $x \in \mathbb{R}^n$. Then either

(a) System

$$f(x) < 0, \quad g(x) \leqq 0, \quad h(x) = 0, x \in S \tag{4.21}$$

has a solution,

or

(b) There exist vectors $\alpha_I \geqslant 0$, $\beta_J \geqq 0$ and γ_K such that for any $x \in S$

$$\sum_{i\in I} \alpha_i f_i(x) + \sum_{j\in J} \beta_j g_j(x) + \sum_{k\in K} \gamma_k h_k(x) \geqq 0, \tag{4.22}$$

but never both.

Proof. Obviously, (a) and (b) cannot be satisfied simultaneously. Assume now that system (4.21) has no solution. Let us set $\varphi(x) = \max_{i\in I} f_i(x)$ and consider the problem of minimizing $\varphi(x)$ subject to $x \in S_1 := \{x \in \mathbb{R}^n \mid g(x) \leqq 0,\ h(x) = 0, x \in S\}$. Obviously, $S_1 \neq \emptyset$ because of the consistency of system (4.20). Since system (4.21) has no solution, we must have

$$\varphi(x) \geqq 0 \quad \forall x \in S_1. \tag{4.23}$$

Let x_0 be the point appearing in the formulation of Theorem 4.2. Observe by the definition of s that $s(x) \geqq 0$ for any $x \in S$. If $\bar{x}$ is a solution of (4.20), then obviously $\bar{x} \in S_1$ and $s(\bar{x}) = 0$. Hence $\min_{x\in S} s(x) = s(\bar{x}) = 0$. Since the function s attains its minimum on S at x_0, $\min_{x\in S} s(x) = s(x_0) = 0$, and hence $x_0 \in S_1$. In addition, $\varphi(x_0) = \max_{i\in I} f_i(x_0) = 0$ since (4.23) holds and $s(x_0) = 0$. Observe also that $I_0 \neq \emptyset$. We have thus proved that $x_0 \in S_1$ and $\varphi(x_0) = 0$. Combining this with (4.23) shows that $\varphi(\cdot)$ attains its minimum on S_1 at x_0 with $\varphi(x_0) = 0$.

By Clarke [4, Theorem 6.1.1, p.228], there exist $\lambda_0 \geqq 0, \beta_j \geqq 0$ $(j \in J_0)$ and $\gamma_k \in \mathbb{R}$ $(k \in K)$, not all zero, such that

$$0 \in \lambda_0 \partial\varphi(x_0) + \sum_{j \in J_0} \beta_j \partial g_j(x_0) + \sum_{k \in K} \gamma_k \partial h_k(x_0) + N_S(x_0). \tag{4.24}$$

(Observe that in our case, $K_0 := K(x_0) = K$.) By condition (CQ), $\lambda_0 \neq 0$ and hence we can take $\lambda_0 = 1$. On the other hand, since φ is defined as the maximum of finitely many functions, we have by Clarke [4, Theorem 2.3.12, p. 47]

$$\partial\varphi(x_0) \subset co\,\{\partial f_i(x_0) : i \in I_0\} \tag{4.25}$$

($I_0 \neq \emptyset$ as we remarked above). Hence, by setting $\alpha_i = 0$ $(i \notin I_0)$ and $\beta_j = 0$ $(j \notin J_0)$ we derive from (4.24) and (4.25) that there exist $\alpha_i \geqq 0$ $(i \in I)$, $\beta_j \geqq 0$ $(j \in J)$ and $\gamma_k \in \mathbb{R}$ $(k \in K)$ such that

$$0 \in \sum_{i \in I} \alpha_i \partial f_i(x_0) + \sum_{j \in J} \beta_j \partial g_j(x_0) + \sum_{k \in K} \gamma_k \partial h_k(x_0) + N_S(x_0) \tag{4.26}$$

where $\alpha_i > 0$ for at least one index $i \in I$. Thus there exist $\xi_i \in \partial f_i(x_0) (i \in I)$, $\bar{\xi}_j \in \partial g_j(x_0) (j \in J)$ and $\bar{\bar{\xi}}_k \in \partial f_k(x_0) (k \in K)$ such that

$$-\left(\sum_{i \in I} \alpha_i \xi_i + \sum_{j \in J} \beta_j \bar{\xi}_j + \sum_{k \in K} \gamma_k \bar{\bar{\xi}}_k \right) \in N_S(x_0).$$

Using the invex-infineness property, we have for any $x \in S$ and a suitable point $\eta = \eta(x) \in T_S(x_0)$

$$\begin{aligned} 0 &\leqq \sum_{i \in I} \alpha_i \langle \xi_i, \eta \rangle + \sum_{j \in J_0} \beta_j \langle \bar{\xi}_j, \eta \rangle + \sum_{k \in K} \gamma_k \langle \bar{\bar{\xi}}_k, \eta \rangle \\ &\leqq \sum_{i \in I} \alpha_i \big[f_i(x) - f_i(x_0) \big] + \sum_{j \in J} \beta_j \big[g_j(x) - g_j(x_0) \big] \\ &\quad + \sum_{k \in K} \gamma_k \big[h_k(x) - h_k(x_0) \big]. \end{aligned}$$

So we have, for any $x \in S$,

$$\begin{aligned} &\sum_{i \in I} \alpha_i f_i(x) + \sum_{j \in J} \beta_j g_j(x) + \sum_{k \in K} \gamma_k h_k(x) \\ &\quad \geqq \sum_{i \in I} \alpha_i f_i(x_0) + \sum_{j \in J} \beta_j g_j(x_0) + \sum_{k \in K} \gamma_k h_k(x_0). \end{aligned}$$

Observing that $\alpha_i f_i(x_0) = 0$ $(i \in I)$, $\beta_j g_j(x_0) = 0$ $(j \in J)$ and $\gamma_k h_k(x_0) = 0$ $(k \in K)$, we obtain (4.22) as desired. □

When g and h are absent in Theorem 4.2, system (4.20) reduces to $f(x) \leqq 0$, $x \in S$, but the consistency of this system is superfluous. Namely, we have

THEOREM 4.3. *(Invex Gordan's Theorem, see [2])Assume that f is a vector-valued map wih locally Lipschitz components f_i $(i \in I)$ and S is a nonempty closed subset of $\mathbb{R}^n$. Assume that $\varphi(x) = max_{i \in I} f_i(x)$ attains its minimum on S at x_0 and f_{I_0} is invex on S at x_0 (in the sense of Definition 3.1) where $I_0 = \{i : f_i(x_0) = \varphi(x_0)\}$. Then either*

(a) System $f(x) < 0, x \in S$ has a solution

or

(b) There exists $\alpha_I \geqslant 0$ such that, for any $x \in S$, $\sum_{i \in I} \alpha_i f_i(x) \geqq 0$, but never both.

Proof. Suppose that (a) does not hold. Since the function φ attains its minimum on S at x_0, it follows from Clarke [4] that

$$0 \in \partial\varphi(x_0) + N_S(x_0).$$

Using (4.25), we again obtain (4.26) with $\alpha_i = 0$ $(i \notin I_0)$ (where ∂g_j and ∂h_k are absent) and hence by invexity of f, for any $x \in S$,

$$\sum_{i \in I} \alpha_i f_i(x) \geqq \sum_{i \in I} \alpha_i f_i(x_0).$$

Since (a) does not hold, $f_i(x_0) = \varphi(x_0) \geqq 0$ for all $i \in I_0$. Thus, for any $x \in S$, $\sum_{i \in I} \alpha_i f_i(x) \geqq 0$. □

REMARK 4.2. Theorem 4.3 was proved in [2] where invexity is understood in the sense of Definition 3.2. Our proof is quite different and simpler than that of [2].

Now we will apply the Gordan type alternative theorem (see Theorem 4.2) to characterizing properly efficient solutions of a vector optimization problem involving invex-infine functions.

Consider the following vector optimization problem :

$$\text{(VOP)} \quad \text{Minimize} \quad f(x) := (f_1(x), \cdots, f_m(x))$$
$$\text{subject to} \quad x \in S_1 := \left\{x \in S : g_j(x) \leqq 0 \ (j \in J), \ h_k(x) = 0 \ (k \in K)\right\}.$$

DEFINITION 4.2. A point $x_0 \in S_1$ is said to be an efficient solution of (VOP) if there does not exist other point $x \in S_1$ such that $f(x) \leqslant f(x_0)$.

DEFINITION 4.3. [13]. A point $x_0 \in S_1$ is said to be a properly efficient point solution of (VOP) if it is an efficient point of (VOP) and if there exists a scalar $M > 0$ such that for each $i \in I$ we have

$$\frac{f_i(x) - f_i(x_0)}{f_{i'}(x_0) - f_{i'}(x)} \leqq M$$

for some $i' \in I$ such that $f_{i'}(x) > f_{i'}(x_0)$ whenever $x \in S_1$ and $f_i(x) < f_i(x_0)$.

THEOREM 4.4. *Assume that* $(f, g_{J_0}; h)$ *is invex-infine on* S *at* x_0, *where* $J_0 = J(x_0) = \{j \in J : g_j(x_0) = 0\}$ *and that condition (CQ) is satisfied at* x_0. *Then* $x_0 \in S_1$ *is a properly efficient solution of (VOP) if and only if there exist* $\lambda_I > 0,\ \mu_{J_0} \geqq 0$ *and* r_K *such that* $x_0 \in S_1$ *solves the following scalar optimization problem:*

$$\text{(VOP)}' \qquad \text{Minimize} \quad s'(x), \quad \text{subject to } x \in S$$

$$\text{where} \qquad s'(x) = \sum_{i\in I} \lambda_i f_i(x) + \sum_{j\in J(x_0)} \mu_j g_j(x) + \sum_{k\in K} r_k h_k(x)$$

Proof. Suppose that $x_0 \in S_1$ is a properly efficient solution of (VOP). Let M be the positive number appearing in Definition 4.3. Then for each $i \in I$ the system

$$\begin{aligned}
&f_i(x) < f_i(x_0),\\
&f_i(x) + Mf_{i'}(x) < f_i(x_0) + Mf_{i'}(x_0), i' \neq i,\\
&g_j(x) \leqq 0\ (j \in J),\\
&h_k(x) = 0\ (k \in K),\\
&x \in S
\end{aligned}$$

has no solution $x \in \mathbb{R}^n$. Indeed, assume to the contrary that for some $i \in I$ this system has a solution x. Then the last three conditions in this system show that $x \in S_1$. Since x_0 is an efficient point and since $f_i(x) < f_i(x_0)$ the index set $\{i' : f_{i'}(x_0) < f_{i'}(x)\}$ must be nonempty. By proper efficiency we must find an element i' of this index set such that

$$\frac{f_i(x) - f_i(x_0)}{f_{i'}(x_0) - f_{i'}(x)} \leqq M$$

or, equivalently, $f_i(x) + Mf_{i'}(x) \geqq f_i(x_0) + Mf_{i'}(x_0)$. This contradicts the second condition in the above system.

Let $i \in I$ be any fixed index. Define

$$\begin{aligned}
&w_i(x) = f_i(x) - f_i(x_0)\\
&w_{i'}(x) = f_i(x) + Mf_{i'}(x) - \big[f_i(x_0) + Mf_{i'}(x_0)\big], i' \neq i
\end{aligned}$$

and $\rho(x) = \max\big\{\max_{i'\in I} w_{i'}(x), \max_{j\in J} g_j(x), \max_{k\in K} |h_k(x)|\big\}$. Then we have $\rho(x_0) = 0$, $I_0 = \big\{i' \in I : w_{i'}(x_0) = \rho(x_0)\big\} = I$, $J_0 := \big\{j \in J : g_j(x_0) = \rho(x_0)\big\} = J(x_0)$ and $K_0 := \{k \in K : |h_k(x_0)| = \rho(x_0)\} = K$. Since $\partial(f_i + Mf_{i'})(x_0) \subset \partial f_i(x_0) + M\partial f_{i'}(x_0)$, and $(f, g_{J_0}; h)$ is invex-infine on S at x_0, then $(w, g_{J_0}; h)$ is also invex-infine on S at x_0 where w is the vector-valued map with components $w_{i'}$ $(i' \in I)$. Since $w_{i'}(x_0) = 0$ $(i' \in I)$, $g_j(x_0) \leqq 0$ $(j \in J)$ and $h_k(x_0) = 0$ $(k \in K)$, then the system

$$w(x) \leqq 0, g(x) \leqq 0, h(x) = 0, x \in S$$

has at least one solution. Thus all the assumptions of Theorem 4.2 are satisfied. Hence, there exist $\alpha^i_{i'} \geqq 0$ $(i' \in I)$, $\beta^i_j \geqq 0$ $(j \in J)$ and $r^i_k \in \mathbb{R}$ $(k \in K)$ such that

$\sum_{i' \in I} \alpha_{i'}^i = 1$ and for any $x \in S$

$$\sum_{i' \in I} \alpha_{i'}^i w_i(x) + \sum_{j \in J} \beta_j^i g_j(x) + \sum_{k \in K} r_k^i h_k(x) \geqq 0.$$

Notice that $\beta_j^i g_j(x_0) = 0$ $(j \in J)$ and $r_k^i h_k(x_0) = 0$ $(k \in K)$ (for this, see the proof of Theorem 4.2). Thus we have for any $x \in S$

$$\begin{aligned} & f_i(x) + M \sum_{i' \neq i} \alpha_{i'}^i f_{i'}(x) + \sum_{j \in J} \beta_j^i g_j(x) + \sum_{k \in K} r_k^i h_k(x) \\ & \geqq f_i(x_0) + M \sum_{i' \neq i} \alpha_{i'}^i f_{i'}(x_0) + \sum_{j \in J} \beta_j^i g_j(x_0) + \sum_{k \in K} r_k^i h_k(x_0) \end{aligned} \tag{4.27}$$

Summing (4.27) over $i \in I$, we see that, for all $x \in S$,

$$\begin{aligned} & \sum_{i' \in I} \lambda_{i'} f_{i'}(x) + \sum_{j \in J} \mu_j g_j(x) + \sum_{k \in K} r_k h_k(x) \\ & \quad \geqq \sum_{i' \in I} \lambda_{i'} f_{i'}(x_0) + \sum_{j \in J} \mu_j g_j(x_0) + \sum_{k \in K} r_k h_k(x_0) \end{aligned} \tag{4.28}$$

and

$$\mu_j g_j(x_0) = 0, \; j \in J \tag{4.29}$$

where

$$\begin{aligned} \lambda_{i'} &= 1 + M \sum_{i \neq i'} \alpha_{i'}^i, \\ \mu_j &= \sum_{i \in J} \beta_j^i, \\ r_k &= \sum_{i \in J} r_k^i. \end{aligned}$$

Hence $x_0 \in S_1$ is an optimal solution of (VOP)′.

Conversely, suppose that $x_0 \in S_1$ is an optimal solution of (VOP)′. Then we have, for any $x \in S_1$,

$$\begin{aligned} \sum_{i \in I} \lambda_i f_i(x_0) &= \sum_{i \in I} \lambda_i f_i(x_0) + \sum_{j \in J(x_0)} \mu_j g_j(x_0) + \sum_{k \in K} r_k h_k(x_0) \\ &\leqq \sum_{i \in I} \lambda_i f_i(x) + \sum_{j \in J(x_0)} \mu_j g_j(x) + \sum_{k \in K} r_k h_k(x) \\ &\leqq \sum_{i \in I} \lambda_i f_i(x). \end{aligned}$$

Thus, by Theorem 1 in [13], x_0 is a properly efficient solution of (VOP). □

Now, we will give a necessary optimality condition for a properly efficient solution of (VOP):

COROLLARY 4.3. *Let $x_0 \in S_1$ be a properly efficient solution of (VOP). Assume that $(f, g_{J_0}; h)$ is invex-infine on S at x_0, where $J_0 = J(x_0) = \{j \in J : g_j(x_0) = 0\}$, and that condition (CQ) is satisfied at x_0. Then there exist $\lambda_I > 0$, $\mu_J \geqq 0$ and r_K such that*

$$0 \in \sum_{i\in I} \lambda_i \partial f_i(x_0) + \sum_{j\in J} \mu_j \partial g_j(x_0) + \sum_{k\in K} r_k \partial h_k(x_0) + N_S(x_0) \text{ and}$$

$$\mu_j g_j(x_0) = 0 (j \in J).$$

Proof. By Theorem 4.4 x_0 must be an optimal solution of Problem (VOP)′. To complete the proof it suffices to apply a result of Clarke [4, p.52, Corollary] and to set $\mu_j = 0$ for $j \notin J(x_0)$. □

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